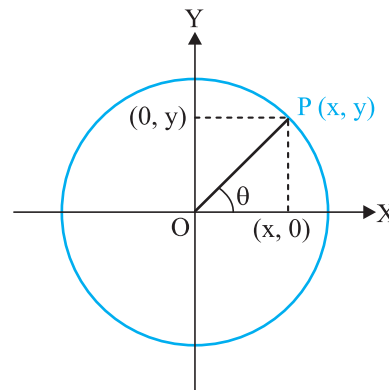


## TRIGONOMETRIC FUNCTIONS OF AN ANGLE

## Section - 2

**2.1** The six trigonometric ratios sine, cosine, tangent, cotangent, secant and cosecant of an angle  $\theta$ ,  $0^\circ < \theta < 90^\circ$  are defined as the ratios of two sides of a right-angled triangle with  $\theta$  as one of acute angle. However, we can also define these trigonometric ratios through a unit circle.

Draw a unit circle (radius = 1 unit) and take any two diameters at right angle as  $X$  and  $Y$  axes. Taking  $OX$  as the initial line, let  $\overline{OP}$  be the radius vector corresponding to an angle  $\theta$ , where  $P$  lies on the unit circle. Let  $(x, y)$  be the coordinates of  $P$ .



Then by definition in Section 1.1.

$$\sin\theta = \frac{p}{h} = \frac{y}{1}, \quad \text{the } y\text{-coordinate of } P$$

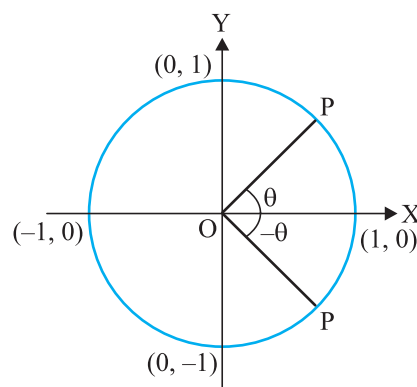
$$\cos\theta = \frac{b}{h} = \frac{x}{1}, \quad \text{the } x\text{-coordinate of } P$$

$$\tan\theta = \frac{p}{b} = \frac{y}{x}, \quad x \neq 0$$

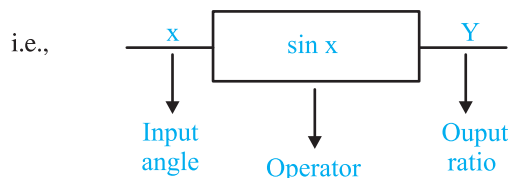
$$\sec\theta = \frac{h}{b} = \frac{1}{x}, \quad x \neq 0, \quad \csc\theta = \frac{h}{p} = \frac{1}{y}, \quad y \neq 0 \quad \text{and} \quad \cot\theta = \frac{b}{p} = \frac{x}{y}.$$

Angle measured anticlockwise from the initial line  $OX$  is positive and angles measured clockwise are considered to be negative.

As we can associate a unique radius vector  $\overline{OP}$  for unique  $P$  with each angle  $\theta$ , we can say ratios of 'x' and 'y' are functions of  $\theta$ . From this, we have an idea that we can study the trigonometric ratios as trigonometric 'function' this holds for all angles.



Therefore, we can study  $y$  or  $f(x) = \sin x$  as trigonometric function.



i.e., for every angle  $\theta$  there exists a unique corresponding ratio *i.e.* output.

Similarly, we can study other trigonometric ratios as trigonometric functions in the same manner.

## 2.2 Signs of Trigonometrical Functions

We have introduced six trigonometric functions. Signs of these ratios depend upon the quadrant in which the terminal side of the angle lies. We always take the length of  $\overline{OP}$  vector is denoted by  $r$  which is always positive.

Thus,  $\sin\theta = \frac{y}{r}$  has the sign of  $y$  and  $\cos\theta = \frac{x}{r}$  has the sign of  $x$ .

The sign of  $\tan\theta$  depends on the signs of  $x$  and  $y$  and similarly the signs of other trigonometric functions can be observed by the signs of  $x$  and  $y$ .

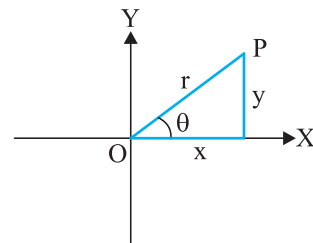
**In First quadrant,** we have

$$x > 0, y > 0$$

$$\therefore \sin\theta = \frac{y}{r} > 0 \quad \cos\theta = \frac{x}{r} > 0, \quad \tan\theta = \frac{y}{x} > 0$$

$$\operatorname{cosec}\theta = \frac{r}{y} > 0, \quad \sec\theta = \frac{r}{x} > 0 \quad \text{and} \quad \cot\theta = \frac{x}{y} > 0$$

Thus, in the first quadrant all trigonometric functions are positive.



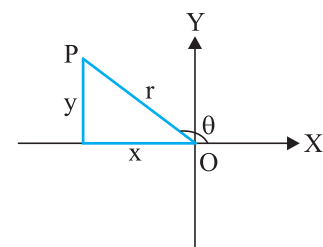
**In second quadrant,** we have

$$x < 0, y > 0$$

$$\therefore \sin\theta = \frac{y}{r} > 0 \quad \cos\theta = \frac{x}{r} < 0, \quad \tan\theta = \frac{y}{x} < 0$$

$$\operatorname{cosec}\theta = \frac{r}{y} > 0, \quad \sec\theta = \frac{r}{x} < 0 \quad \text{and} \quad \cot\theta = \frac{x}{y} < 0$$

Thus, in the second quadrant all trigonometric functions are negative other than sine and cosecant.



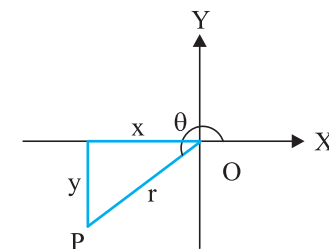
**In third quadrant,** we have

$$x < 0, y < 0$$

$$\therefore \sin\theta = \frac{y}{r} < 0 \quad \cos\theta = \frac{x}{r} < 0, \quad \tan\theta = \frac{y}{x} > 0$$

$$\operatorname{cosec}\theta = \frac{r}{y} < 0, \quad \sec\theta = \frac{r}{x} < 0 \quad \text{and} \quad \cot\theta = \frac{x}{y} > 0$$

Thus, in the third quadrant all trigonometric functions are negative other than tangent and cotangent.



**In fourth quadrant,** we have

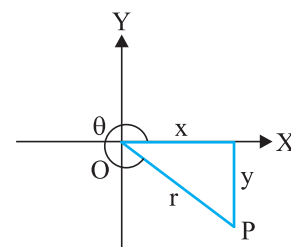
$$x > 0, y < 0$$

$$\therefore \sin\theta = \frac{y}{r} < 0 \quad \cos\theta = \frac{x}{r} > 0, \quad \tan\theta = \frac{y}{x} < 0$$

$$\operatorname{cosec}\theta = \frac{r}{y} < 0 \quad \sec\theta = \frac{r}{x} > 0 \quad \text{and} \quad \cot\theta = \frac{x}{y} < 0$$

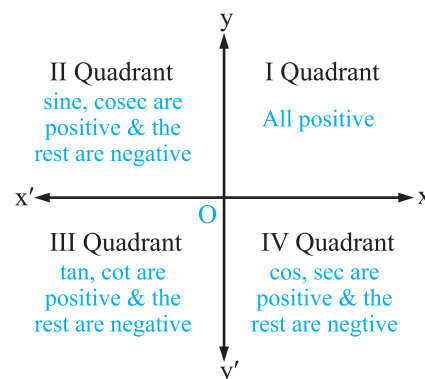
Thus, in the fourth quadrant all trigonometric functions are negative other than cosine and secant.

It follows from the above discussion that the signs of the trigonometric ratios in different quadrants are as follows:



### 2.3 Sign of T-ratios in four Quadrants:

- If revolving ray lies in Q-I,  $x$  and  $y$  are positive, hence all T-ratios are positive.
- If revolving ray lies in Q-II,  $x$  is negative and  $y$  is positive, hence only  $\sin\theta$  and  $\operatorname{cosec}\theta$  are positive.
- If revolving ray lies in Q-III,  $x$  is negative and  $y$  is negative, hence only  $\tan\theta$  and  $\cot\theta$  positive.
- If revolving ray lies in Q-IV,  $x$  is positive and  $y$  is negative, hence only  $\cos\theta$  and  $\sec\theta$  are positive.



### 2.4. Graph and Properties of Trigonometric Functions [ $n \in \mathbb{I}$ ]

#### 1. $y = \sin\theta$ or $\sin x$

As we have explained all trigonometric ratios are functions (i.e., relations between angle and ratio) which implies that each trigonometric function must possess pictorial representation i.e. graph.

Now, for graph of trigonometric  $y = \sin x$ , we have to observe nature of  $y = \sin x$  in different quadrants.

#### In 1st quadrant :

As  $x$  varies from 0 to  $\frac{\pi}{2}$  then corresponding ratio of  $y = \sin x$  is positive and increases from 0 to 1.

#### In 2nd quadrant :

As  $x$  varies from  $\frac{\pi}{2}$  to  $\pi$  then corresponding ratio of  $y = \sin x$  is positive and decreases from 1 to 0.

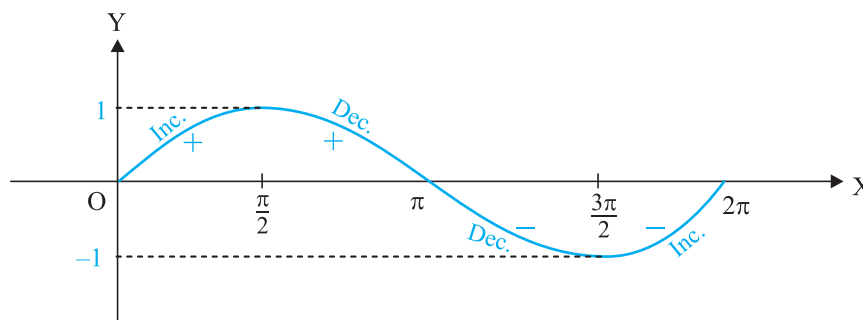
#### In 3rd quadrant :

As  $x$  varies from  $\pi$  to  $\frac{3\pi}{2}$  then corresponding ratio of  $y = \sin x$  is negative and decreases from 0 to  $-1$ .

#### In 4th quadrant :

As  $\theta$  varies from  $\frac{3\pi}{2}$  to  $2\pi$  then corresponding ratio of  $y = \sin x$  is negative and increases from  $-1$  to 0.

Now, from the above discussion, we have the graph of  $y = \sin x$  in 0 to  $2\pi$  as

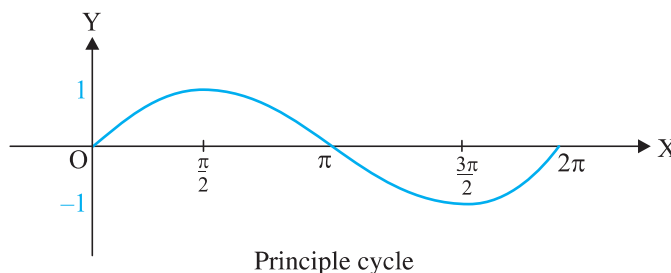


If angle is increased further from  $2\pi$  then we observe that its ratio i.e. output starts repeating.

Which implies  $y = \sin x$  is periodic function with fundamental period  $= 2\pi$ .

**Properties of  $y = \sin x$  :**

- (i) Domain of  $y = \sin x$  is  $x \in R$
- (ii) Range of  $y = \sin x$  is  $y \in [-1, 1]$
- (iii) It is periodic function with fundamental period of  $2\pi$ .



- (iv) variation of  $y = \sin x$  is

$$y \in [-1, 1] \Rightarrow -1 \leq \sin x \leq 1 \quad \Rightarrow \quad y_{\max} = 1 \text{ and } y_{\min} = -1$$

- (v) Variation of  $y = A \sin(mx)$

$$y \in [-A, A] \Rightarrow -A \leq A \sin(mx) \leq A$$

$$\Rightarrow A \sin(mx) \text{ can never be greater than } A \text{ or less than } -A$$

$$\Rightarrow y_{\max} = A \text{ and } y_{\min} = -A$$

- (vi) Period of  $A \sin mx$  is  $T = \frac{2\pi}{m}$

**2.  $y = \cos \theta$  or  $\cos x$** **In 1st quadrant :**

As  $x$  varies from 0 to  $\frac{\pi}{2}$  then corresponding ratio of  $y = \cos x$  is positive and decreases from 1 to 0.

**In 2nd quadrant :**

As  $x$  varies from  $\frac{\pi}{2}$  to  $\pi$  then corresponding ratio of  $y = \cos x$  is negative and decreases from 0 to  $-1$ .

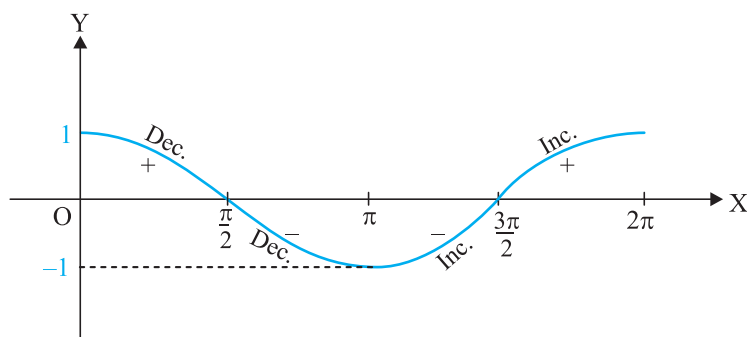
**In 3rd quadrant :**

As  $x$  varies from  $\pi$  to  $\frac{3\pi}{2}$  then corresponding ratio of  $y = \cos x$  is negative and increases from  $-1$  to 0.

**In 4th quadrant :**

As  $x$  varies from  $\frac{3\pi}{2}$  to  $2\pi$  then corresponding ratio of  $y = \cos x$  is positive and increases from 0 to 1.

Now, from the above discussion, we have the graph of  $y = \cos x$  in 0 to  $2\pi$  as



If angle is increased further from  $2\pi$  then we observe that its ratio i.e. output starts repeating.

Which implies  $y = \cos x$  is periodic function with fundamental period  $= 2\pi$ .

**Properties of  $y = \cos x$  :**

- (i) Domain of  $y = \cos x$  is  $x \in R$
- (ii) Range of  $y = \cos x$  is  $y \in [-1, 1]$
- (iii) It is periodic function with fundamental period of  $2\pi$ .

- (iv) variation of  $y = \cos x$   
 $y \in [-1, 1]$

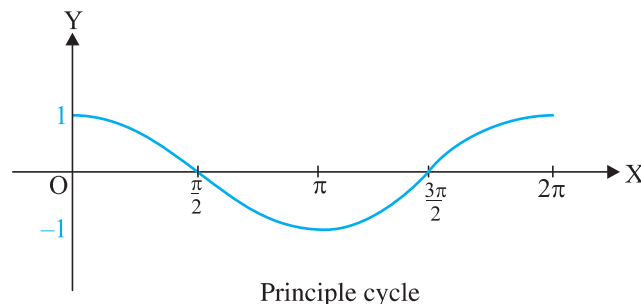
$$\Rightarrow -1 \leq \cos x \leq 1 \quad \Rightarrow \quad y_{\max} = 1 \text{ and } y_{\min} = -1$$

- (v) Variation of  $y = A \cos (mx)$

$$\text{is } y \in [-A, A] \Rightarrow -A \leq A \cos (mx) \leq A$$

$$\Rightarrow A \cos (mx) \text{ can never be greater than } A \text{ or less than } -A \quad \Rightarrow \quad y_{\max} = A \text{ and } y_{\min} = -A$$

- (vi) Period of  $A \cos mx$  is  $T = 2\pi/m$

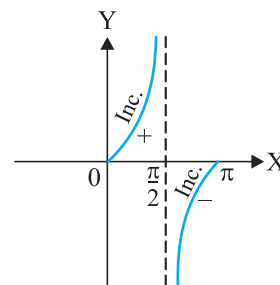
**3.  $y = \tan \theta$  or  $\tan x$** **In 1st quadrant :**

As  $x$  varies from  $0$  to  $\frac{\pi}{2}$  then corresponding ratio of  $y = \tan x$  is positive and increases from  $0$  to  $\infty$  and at  $x = \frac{\pi}{2}$   $y = \tan x$  is not defined.

**In 2nd quadrant :**

As  $x$  varies from  $\frac{\pi}{2}$  to  $\pi$  then corresponding ratio of  $y = \tan x$  is negative and increases from  $-\infty$  to  $0$ .

Now, from the above discussion, we have the graph of  $y = \tan x$  in  $0$  to  $\pi$  as



If angle is increased further from  $\pi$  then we observe that its ratio i.e. output starts repeating.

Which implies  $y = \tan x$  is periodic function with fundamental period  $= \pi$ .

**Properties of  $y = \tan x$  :**

- (i) Domain of  $y = \tan x$  is  $x \in R - \left\{ (2n+1) \frac{\pi}{2} : n \in I \right\}$

- (ii) Range of  $y = \tan x$  is  $y \in (-\infty, \infty)$

- (iii) It is periodic function with fundamental period of  $\pi$ .

- (iv) variation of  $y = \tan x$

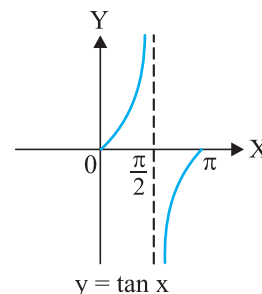
$$y \in (-\infty, \infty) \Rightarrow -\infty < \tan (x) < \infty$$

$$\Rightarrow y_{\max} = \infty \text{ and } y_{\min} = -\infty$$

- (v) Variation of  $y = A \tan (mx)$  is  $y \in (-\infty, \infty)$

$$\Rightarrow -\infty < A \tan (mx) < \infty \Rightarrow y_{\max} = \infty \text{ and } y_{\min} = -\infty$$

- (vi) Period of  $A \tan (mx)$  is  $T = \pi/m$

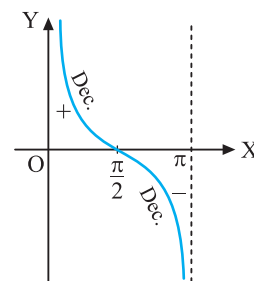


4.  $y = \cot(x)$ Properties of  $y = \cot(x)$ (i) Domain of the  $y = \cot(x)$  is  $x \in R - \{n\pi : n \in I\}$ (ii) Range of the  $y = \cot(x)$  is  $y \in (-\infty, \infty)$ (iii) It is a periodic function with period of  $\pi$ .(iv) Variation of  $y = \tan x$ 

$$y \in (-\infty, \infty) \Rightarrow -\infty < \cot(mx) < \infty \Rightarrow y_{\max} = \infty \text{ and } y_{\min} = -\infty$$

(v) Variation of  $y = A \cot(mx)$ 

$$\text{as } y \in (-\infty, \infty) \Rightarrow -\infty < A \cot(mx) < \infty \Rightarrow y_{\max} = \infty \text{ and } y_{\min} = -\infty$$

(vi) Period of  $A \cot(mx)$  is  $T = \pi/m$ Principal cycle of  $A \cot x$ 5.  $y = \operatorname{cosec}(x)$ Properties of  $y = \operatorname{cosec}(x)$ (i) Domain of the  $y = \operatorname{cosec}(x)$  is  $x \in R - \{n\pi : n \in I\}$ (ii) Range of the  $y = \operatorname{cosec}(x)$  is  $y \in (-\infty, -1] \cup [1, \infty)$ (iii) It is periodic function with period of  $2\pi$ (iv) variation of  $y = \operatorname{cosec} x$ 

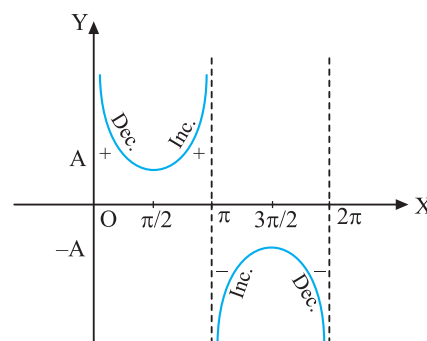
$$y \in (-\infty, -1] \cup [1, \infty)$$

$$\Rightarrow y_{\max} = \infty \text{ and } y_{\min} = -\infty$$

(v) Variation of  $y = A \operatorname{cosec}(mx)$  as  $y \in (-\infty, -A] \cup [A, \infty)$ 

$$\Rightarrow A \operatorname{cosec}(mx) \text{ can be greater than } A \text{ or less than } -A$$

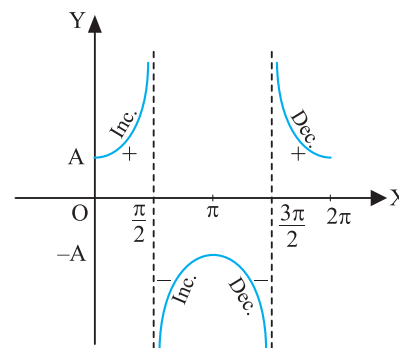
$$\Rightarrow y_{\max} = \infty \text{ and } y_{\min} = -\infty$$

(vi) Period of  $A \operatorname{cosec}(mx)$  is  $T = 2\pi/m$ Principal cycle of  $A \operatorname{cosec} x$ 6.  $y = \sec(x)$ Properties of  $y = \sec(x)$ (i) Domain of the  $y = \sec(x)$  is  $x \in R - \left\{(2n+1)\frac{\pi}{2} : n \in I\right\}$ (ii) Range of the  $y = \sec(x)$  is  $y \in (-\infty, -1] \cup [1, \infty)$ (iii) It is periodic function with period of  $2\pi$ (iv) variation of  $y = \sec x$ 

$$y \in (-\infty, -1] \cup [1, \infty) \Rightarrow y_{\max} = \infty \text{ and } y_{\min} = -\infty$$

(iv) Variation of  $y = A \sec(mx)$  as  $y \in (-\infty, -A] \cup [A, \infty)$ 

$$\Rightarrow A \sec(mx) \text{ can be greater than } A \text{ or less than } -A \Rightarrow y_{\max} = \infty \text{ and } y_{\min} = -\infty$$

(vi) Period of  $A \sec(mx)$  is  $T = 2\pi/m$ Principal cycle of  $A \sec x$

## 2.5. Trigonometric Ratios of some Standard Acute Angles

The values or trigonometric ratios of standard acute angles are put in the following tabular form for ready reference.

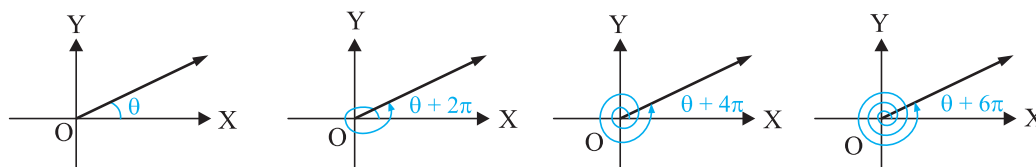
**Trigonometric Ratios of Standard Angles**

| Degree       | 0 | 30           | 45           | 60           | 90       | 120          | 135           | 150           | 180   | 270      |
|--------------|---|--------------|--------------|--------------|----------|--------------|---------------|---------------|-------|----------|
| Radian       | 0 | $\pi/6$      | $\pi/4$      | $\pi/3$      | $\pi/2$  | $2\pi/3$     | $3\pi/4$      | $5\pi/6$      | $\pi$ | $3\pi/2$ |
| $\sin\theta$ | 0 | $1/2$        | $1/\sqrt{2}$ | $\sqrt{3}/2$ | 1        | $\sqrt{3}/2$ | $1/\sqrt{2}$  | $1/2$         | 0     | -1       |
| $\cos\theta$ | 1 | $\sqrt{3}/2$ | $1/\sqrt{2}$ | $1/2$        | 0        | $-1/2$       | $-1/\sqrt{2}$ | $-\sqrt{3}/2$ | -1    | 0        |
| $\tan\theta$ | 0 | $1/\sqrt{3}$ | 1            | $\sqrt{3}$   | $\infty$ | $-\sqrt{3}$  | -1            | $-1/\sqrt{3}$ | 0     | $\infty$ |

## 2.6. Ratios of Allied Angles

### T-ratios for $2n\pi + \theta$ :

Revolving ray assumes the same position for  $\theta, 2\pi + \theta, 4\pi + \theta, 6\pi + \theta$ .



**Note:** That T-ratios will be same for  $\theta, 2\pi + \theta, 4\pi + \theta, 6\pi + \theta$ , or one can say same for  $\theta$  and  $\theta + 2n\pi, n \in I$

$$\Rightarrow \sin(2n\pi + \theta) = \sin \theta$$

$$\cos(2n\pi + \theta) = \cos \theta$$

$$\tan(2n\pi + \theta) = \tan \theta$$

Hence adding and subtracting a multiple of  $2\pi$  in an angle does not change the value of T-ratio.

### T-Ratios for $\frac{\pi}{2} - \theta$ :

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$$

$$\operatorname{cosec}\left(\frac{\pi}{2} - \theta\right) = \sec \theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$$

$$\sec\left(\frac{\pi}{2} - \theta\right) = \operatorname{cosec} \theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot \theta$$

$$\cot\left(\frac{\pi}{2} - \theta\right) = \tan \theta$$

### T-Ratios for $\frac{\pi}{2} + \theta$ :

$$\sin\left(\frac{\pi}{2} + \theta\right) = +\cos \theta$$

$$\operatorname{cosec}\left(\frac{\pi}{2} + \theta\right) = \sec \theta$$

$$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin \theta$$

$$\sec\left(\frac{\pi}{2} + \theta\right) = -\operatorname{cosec} \theta$$

$$\tan\left(\frac{\pi}{2} + \theta\right) = -\cot \theta$$

$$\cot\left(\frac{\pi}{2} + \theta\right) = -\tan \theta$$

**T-Ratios for  $\pi - \theta$  :**

$$\sin(\pi - \theta) = \sin \theta$$

$$\tan(\pi - \theta) = -\tan \theta$$

$$\sec(\pi - \theta) = -\sec \theta$$

$$\cos(\pi - \theta) = -\cos \theta$$

$$\cot(\pi - \theta) = -\cot \theta$$

$$\operatorname{cosec}(\pi - \theta) = \operatorname{cosec} \theta$$

**T-Ratios for  $\pi + \theta$  :**

$$\sin(\pi + \theta) = -\sin \theta$$

$$\tan(\pi + \theta) = \tan \theta$$

$$\sec(\pi + \theta) = -\sec \theta$$

$$\cos(\pi + \theta) = -\cos \theta$$

$$\cot(\pi + \theta) = \cot \theta$$

$$\operatorname{cosec}(\pi + \theta) = -\operatorname{cosec} \theta$$

**T-Ratios for  $\frac{3\pi}{2} - \theta$  :**

$$\sin\left(\frac{3\pi}{2} - \theta\right) = -\cos \theta$$

$$\tan\left(\frac{3\pi}{2} - \theta\right) = \cot \theta$$

$$\sec\left(\frac{3\pi}{2} - \theta\right) = -\operatorname{cosec} \theta$$

$$\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin \theta$$

$$\cot\left(\frac{3\pi}{2} - \theta\right) = \tan \theta$$

$$\operatorname{cosec}\left(\frac{3\pi}{2} - \theta\right) = -\sec \theta$$

**T-Ratios for  $\frac{3\pi}{2} + \theta$  :**

$$\sin\left(\frac{3\pi}{2} + \theta\right) = -\cos \theta$$

$$\tan\left(\frac{3\pi}{2} + \theta\right) = -\cot \theta$$

$$\sec\left(\frac{3\pi}{2} + \theta\right) = \operatorname{cosec} \theta$$

$$\cos\left(\frac{3\pi}{2} + \theta\right) = +\sin \theta$$

$$\cot\left(\frac{3\pi}{2} + \theta\right) = -\tan \theta$$

$$\operatorname{cosec}\left(\frac{3\pi}{2} + \theta\right) = -\sec \theta$$

**T-Ratios for  $2\pi - \theta$  :**

$$\sin(2\pi - \theta) = -\sin \theta$$

$$\tan(2\pi - \theta) = -\tan \theta$$

$$\sec(2\pi - \theta) = \sec \theta$$

$$\cos(2\pi - \theta) = \cos \theta$$

$$\cot(2\pi - \theta) = -\cot \theta$$

$$\operatorname{cosec}(2\pi - \theta) = -\operatorname{cosec} \theta$$

**T-Ratios for  $2\pi + \theta$  :**

$$\sin(2\pi + \theta) = \sin \theta$$

$$\tan(2\pi + \theta) = \tan \theta$$

$$\sec(2\pi + \theta) = \sec \theta$$

$$\cos(2\pi + \theta) = \cos \theta$$

$$\cot(2\pi + \theta) = \cot \theta$$

$$\operatorname{cosec}(2\pi + \theta) = \operatorname{cosec} \theta$$

**T-Ratios for negative  $(-\theta)$  :**

$$\sin(-\theta) = -\sin \theta$$

$$\tan(-\theta) = -\tan \theta$$

$$\operatorname{cosec}(-\theta) = -\operatorname{cosec} \theta$$

$$\cos(-\theta) = \cos \theta$$

$$\cot(-\theta) = -\cot \theta$$

$$\sec(-\theta) = \sec \theta$$

**Illustrating the above concepts :**

- $\sin(1050^\circ) = \sin(1080^\circ - 30^\circ) = \sin(6\pi - \pi/6) = \sin(-\pi/6) = -\sin \pi/6 = -1/2$
- $\sin(120^\circ) = \sin 2\pi/3 = \sin(\pi - \pi/3) = \sin \pi/3 = \sqrt{3}/2$
- $\cos(120^\circ) = \cos 2\pi/3 = \cos(\pi - \pi/3) = -\cos \pi/3 = -1/2$
- $\tan(120^\circ) = \tan 2\pi/3 = \tan(\pi - \pi/3) = -\tan \pi/3 = -\sqrt{3}$
- $\sin(135^\circ) = \sin 3\pi/4 = \sin(\pi - \pi/4) = \sin \pi/4 = 1/\sqrt{2}$
- $\sin 9\pi/4 = \sin(2\pi + \pi/4) = \sin \pi/4 = 1/\sqrt{2}$
- $\sin 11\pi/3 = \sin(4\pi - \pi/3) = -\sin \pi/3 = -\sqrt{3}/2$
- $\cos 31\pi/6 = \cos(5\pi + \pi/6) = \cos(\pi + \pi/6) = -\cos \pi/6 = -\sqrt{3}/2$
- $\tan 41\pi/6 = \tan(7\pi - \pi/6) = \tan(\pi - \pi/6) = -\tan \pi/6 = -1/\sqrt{3}$